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SOCIAL WELFARE AS WEIGHTED SUM OF INCOMES

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Abstract

Welfare judgements on distribution of income has often been made on the basis of weighted sum of individual incomes. Scholars proposed various axioms to determine the weights. The set of weights proposed by Chakrabarty (1982) led to a Social welfare function (SWF) involving the normalised Theil's Entropy Measure. We have proposed a few alternative sets of weights here which lead to SWFs involving the Theil's measure as such and a new measure of inequality. An alternative formulation leads to another pair of SWFs involving the Variance of Income Power. These SWFs are seen to have certain interesting properties and interpretations.

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Social Welfare as Weighted Sum of Incomes

1. INTRODUCTION

Among all inequality measures, Gini Coefficient is the most thoroughly scrutinised and popular in empirical applications. Following its criticism by Atkinson (1970) the welfare implications of Gini Co-efficient have been extensively analysed by Newbery (1970), Sheshinsky (1972), Kats (1972), Dasgupta, Sen and Starrett (1973), Rothschild and Stiglitz (1973), Sen (1973, 1974) and Kakwani (1980, 1981, 1985), among others. Normative aspects of other measures, including those by Theil, have not yet received similar scholarly attention. Sen regarded Theil's measure to be arbitrary. *"But the fact remains that it is an arbitrary formula, and the average of the logarithms of reciprocals of income shares weighted by income shares is not a measure that is exactly overflowing with intuitive sense."* (Sen 1973, pp 35-36).

Chakravarty (1982) made a complete axiomatisation of a social welfare function (SWF) that will rank all possible income profiles, with the same number of earners and the same mean income, in the same way as the normalised Theil's measure.

This paper attempts to extract *intuitive sense* out of the axioms proposed by Chakravarty (1982). His axiomatic set up is similar to those of Sen (1974) and Kakwani (1980, 1981). These scholars attempted to measure social welfare as weighted sum of individual incomes. The set of weights has been chosen through a set of axioms conforming to commonly held notions about welfare. The set of weights suggested by Chakravarty (1982) has been scrutinised in section 3 to bring out the aspects of relative deprivation it captures. In Section 4, we propose three alternative sets of weights within this axiomatic set up. One of these leads to a social welfare function which is a negative transformation of Theil's Measure as such, not normalised. The other two involve a new measure of inequality having some similarity with the Gini Coefficient and conforming to many of the desirable properties. In section 5, we present an alternative formulation and derive a pair of social welfare functions which are negative transformations of Variance of Income Power, normalised and not normalised. The set of weights corresponding to inequality measures not normalised are shown to have certain interesting properties and interpretations in section 6.

2. THE AXIOMATIC APPROACH

Let $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n)$ be the given income profile of the community of **n** individuals with mean income $\boldsymbol{\mu}$ so that[2]

$$\mathbf{n}\boldsymbol{\mu} = \Sigma\, \mathbf{y}_i \qquad \text{...........(1)}$$

The mean income $\boldsymbol{\mu} > \mathbf{0}$, is assumed to be finite and same for all income profiles under comparison.

Like Sen (1974) and Kakwani (1980, 1981), Chakravarty (1982) considers only those social welfare functions which are weighted sum of the individual income shares or incomes.

$$\mathbf{W(Y)} = \Sigma\, \mathbf{w}_i\mathbf{y}_i \qquad \text{................(2)}$$

where $\mathbf{w}_i$ is the weight attached to income $\mathbf{y}_i$ and is a function of the entire income profile, not of $\mathbf{y}_i$ alone.

In the literature $\mathbf{w}_i$**'s** have been assumed to be non-negative. The restriction appears to be somewhat arbitrary and hence not insisted upon here.

Chakrabarty (1982) replaced Sen's Rank Order Axiom R by axiom D while retaining his other axioms M (Independent Monotonocity)) and E (Monotonic Equity).

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[2] All summations here have range between **i=1** to **i=n**. Since there is no chance of confusion the range is ommitted for convenience in printing.

The welfare function corresponding to the set of weights (18) would be

$$\mathbf{W_6(\log Y) = A_6\Sigma\,(n\log M - \log y_i)\log y_i}$$
$$\mathbf{= A_6 n[(n - 1)(\log M)^2 - H^2]}$$

If each $\mathbf{\log y_i = \log M}$, then $\mathbf{H^2 = 0}$ and

$$\mathbf{W_6(\log Y) = A_6 n(n - 1)(\log M)^2 = \log M}, \text{ by Axiom } N^*,$$

implying $\mathbf{A_6 = 1/[n(n - 1)\log M]}$ and

$$\mathbf{W_6(\log Y) = \log M[1 - H^2/\{(n - 1)(\log M)^2\}]} \qquad \dots \dots \dots (22)$$

The function $\mathbf{W_5(\log Y)}$ is a decreasing function of the Variance of Income Power while $\mathbf{W_6(\log Y)}$ is that of the normalised Variance of Income Power. The other two sets of weights (9) and (10) do not correspond to any popular measure of inequality.

6. GENESIS OF THE WELFARE MEASURES

Pareto Principle and Population Repluication

Let us confine our attention to the six welfare measures derived by choosing six different set of weights conforming to the axioms **M, E, D** and **N** viz. $\mathbf{W_1}$, $\mathbf{W_2}$, $\mathbf{W_3}$, $\mathbf{W_4}$, $\mathbf{W_5}$ and $\mathbf{W_6}$. The weight attached to income $\mathbf{y_i}$ (or income power $\mathbf{\log y_i}$) in getting the weighted sum $\Sigma\,\mathbf{w_i y_i}$ (or $\Sigma\,\mathbf{w_i \log y_i}$) is proportional to the shortfall of income power realised from the income power *aspired for.* This makes the sense of relative deprivation proportional to the gap between the *desired position* and the *position realised.* The position realised corresponds to the realised income power $\mathbf{\log y_i}$. Different choice of the desired position leads to different welfare measures. Some of these are decreasing functions of some measures of inequality in popular use.

If the desired position corresponds to having all the income of the society leaving nothing for others the corresponding welfare function turns out to be a function of some measure of inequality normalised by its maximum attainable value such as

$$\mathbf{W_1(Y) = \mu[1 - T/\log n]}$$
$$\mathbf{W_4(Y) = \mu[1 - CH/\{(n\text{-}1)\log M\}]}$$
$$\mathbf{W_6(\log Y) = \log M[1 - H^2/\{(n\text{-}1)(\log M)^2\}]}$$

Axiom D (Weight Differential Information)

If $W_i > W_j$ then the weight differential $(w_j - w_i)$ depends only on the ratio of the incomes (y_i/y_j) where W_i is the level of welfare of the individual i in the income profile Y .

This axiom makes the weight differential depend only on the relative magnitudes of incomes y_i and y_j and thus captures some aspect of relative deprivation. The sense of relative deprivation contained in his set of weights have interesting interpretations to which we shall turn shortly.

3. Social Welfare Function Implied by Chakrabarty (1982)

The set of weights conforming to axioms **M, E** and **D** suggested by Chakravarty (1982), is,

$$w_{1i} = A_1 \log (n\mu/y_i) \qquad \text{..........(7)}$$

where A_1 is the constant of proportionality to be determined.

Accepting the set of weights w_{1i} the social welfare function becomes[3],

$$W_1(Y) = A_1 \Sigma\, y_i \log(n\mu/y_i)$$

$$= A_1 n\mu[\log n - T]$$

where

$$T = (1/n)\Sigma\, [(y_i/\mu)\log (y_i/\mu)]$$

is Theil's Entropy Measure.

Accepting Sen's Normalisation Axiom N, value of A_1 can be determined.

Axiom N (Normalisation)

If all incomes are equal then W(Y) = μ, where μ is the mean income of the profile.

If each $y_i = \mu$ then $T = 0$ and $W_1(Y) = A_1 n\mu \log n = \mu$, by Axiom N,

implying $A_1 = (1/n\log n)$ and

$$W_1(Y) = \mu[1 - T/\log n] \qquad \text{... ... ... (8)}$$

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[3] Details of this and other calculations are shown in the appendix.

Equation (8) shows that given the set of axioms and fixed **μ** and **n** the social welfare function **$W_1(Y)$** is a decreasing function of normalised Entropy measure **T/log n.**

Let us now scrutinise this set of weights by rewriting this as

$$\mathbf{w_{1i} = (1/n\log n)(\log n\mu - \log y_i)} \qquad \text{... ... ... (9)}$$

Sense of Relative Deprivation
Contained in w_{1i}

A closer look at the set of weights in (9) reveals the sense of relative deprivation contained in it. We know **$n\mu$** is the total income of the community and **$\log n\mu$** is the maximum income power the individual can think of realising if he can have all the income **$n\mu$**, leaving nothing for others in the community. His sense of deprivation is, thus, proportional to the gap between this maximum possible income power and the income power he could realise. Each person in the profile feels deprived for not having the entire income **$n\mu$** of the profile. Is this a realistic scenario?

Perhaps, there is a wide agreement with Runciman (1966, p 10) that a person is to be considered relatively deprived of income **y** when

- he does not have income **y**,

- he sees some other person or persons, which may include himself at some previous or expected time, as having income **y**, and

- he sees it as feasible that he should have income **y**.

It appears unlikely that in any real situation the individual will see some other person as having the entire income of the community; neither it appears likely that he will see it as feasible that he should have the entire income **$n\mu$**, while all others have no income. It may, therefore, be more realistic if his ambition can be lowered.

Moreover, the normalised measure **T/log n** fails to conform to the Principle of Population Replication[4] (PPR), though **T** dces so. Consequently, $\mathbf{W_1(Y)}$ would fail to conform to PPR.

4. Alternative Sets of Weights

The set of weights $\mathbf{w_{1i}}$ is not the only set conforming to axioms **M, E** and **D**. There are in fact many others. Consider any arbitrary level of income $\mathbf{y^* > 0}$ and let the weights be proportional to the gap $\mathbf{(\log y^* - \log y_i)}$ so that

$$\mathbf{w^*_i = A^*(\log y^* - \log y_i)}$$
$$\mathbf{= A^* \log (y^*/y_i)}$$

It can be easily verified that the set of weights $\mathbf{w^*_i}$ conforms to axioms **M, E** and **D**.

The motivation behind choosing $\mathbf{y^* = n\mu}$ is perhaps, to ensure that each of the weights is non-negative so that the resulting welfare function is increasing in each individual income and thus conform to the Pareto Principle[5] (PP). We shall have more to say on this as we proceed. Here we shall examine the implication of three other alternative sets of weights conforming to axioms **M, E, D** and **N**.

Alternative Set of Weights 1

By choosing $\mathbf{y^* = n\mu}$ the resulting welfare function $\mathbf{W_1(Y)}$ fails to conform to PPR. It will conform to PPR if **T/log n** is replaced by **T** or equivalently **log n** is replaced by $\mathbf{1 = \overline{\log} e}$ where **e** is the base of the Naperian logarithm. This can be achieved by persuading the individual to aspire not for the entire income $\mathbf{n\mu}$ but a lower multiple of $\mathbf{\mu}$, viz. $\mathbf{\mu e}$. The weight attached to income $\mathbf{y_i}$ would, then, become

$$\mathbf{w_{2i} = A_2(\log \mu e - \log y_i)} \qquad \dots \dots \dots (10)$$

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[4] This principle demands that the measure should not depend on the number of individuals receiving income. If one measures welfare in a particular society with **n** individuals in it and then merge another identical one, he gets a society of **2n** individuals with the same proportion of individuals earning any given income. If the measured level of welfare remains same for any such replication of the society then the welfare measure is said to conform to the Principle of Population Replication (PPR).

[5] This principle demands that the welfare function should be an increasing function of every income. If any of the incomes is raised the level of welfare should rise.

The social welfare function implied by this set of weights is

$$W_2(Y) = A_2\Sigma\ (\log \mu e - \log y_i)y_i$$
$$= A_2 n\mu(1 - T)$$

If each $y_i = \mu$ then $T = 0$ and $W_2(Y) = A_2 n\mu = \mu$ by Axiom N, implying $A_2 = (1/n)$ and

$$W_2(Y) = \mu(1 - T) \qquad \text{... ... ... (11)}$$

which is a negative transformation of Theil's Measure as such, not normalised. Consequently, the implied social welfare function $W_2(Y)$ conforms to the Principle of Population Replication. We shall see subsequently that this is achieved at the cost of violating another *desirable* property.

Alternative Set of Weights 2

The purpose of replacing **nμ** by **μe** in (10) has been to replace **T/log n** by **T** in (11) so that the resulting welfare function conforms to PPR and the sense of relative deprivation contained in the set of weights becomes somewhat free from the deficiencies noted. As such the income level **μe** has no other special significance. The purpose can be achieved by replacing **nμ** by other multiples of **e** which are less than **nμ**. The income level **Me** is one such where **M** is the geometric mean of incomes.

$$n \log M = \Sigma \log y_i \qquad \text{... ... ... (12)}$$

Unlike **μe** the income level **Me** has some significance. The measure of inequality Variance of Income Power, H^2 violates Pigou-Dalton Axiom[6] (PDA) when income transfers are made beyond this income level.

$$H^2 = (1/n)\ \Sigma\ (\log y_i - \log M)^2$$

What is the consequence on the set of weights and the social welfare function if the individual aspires for an income level **Me** instead of **μe** ? His sense of deprivation

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[6] This axiom demands that transfer of income from poor to rich should (increase inequality and) decrease welfare while transfers from rich to poor should increase welfare.

would now be proportional to the gap **($\log Me - \log y_i$)** suggesting the set of weights

$$w_{3i} = A_3(\log Me - \log y_i) \quad \text{... ... ... (13)}$$

The implied social welfare function would be,

$$W_3(Y) = A_3\Sigma\,(1 + \log M - \log y_i)y_i$$
$$= A_3 n\mu[1 - (1/n\mu)\Sigma\,(\log y_i - \log M)y_i]$$
$$= A_3 n\mu[1 - (1/\mu)Cov(\log y_i,\, y_i)]$$

If each $y_i = \mu$, then **$Cov(\log y_i, y_i) = 0$.**

Normalising by Sen's axiom N we get $A_3 = (1/n)$ and

$$W_3(Y) = \mu[1 - (1/\mu)Cov(\log y_i\,, y_i)]$$
$$= \mu[1 - CH] \quad \text{... ... ... (14)}$$

where

$$CH = (1/\mu)Cov(\log y_i, y_i) \quad \text{... ... ... (15)}$$

A New Measure of Inequality

This suggests **CH** as a contending measure of inequality. It is easily seen that the measure is mean independent, symmetric and has some similarity in appearance to our Gini coefficient. The Gini coefficient can be expressed as a multiple of covariance between the incomes and their ranks, **$G = (2/n\mu)Cov(i, y_i)$**, when incomes are arranged in ascending order to have, $y_i \leq y_{i+1}$.

The measure **CH** is also a multiple of covariance between the income and the income power. The income power, **log y**, can also be thought of as a different way of *ranking* the incomes.

When the distribution is perfectly egalitarian, each $y_i = \mu$ the measure **CH** assumes its minimum value **0**. The upper limit cannot be obtained because when all income accrue in one hand, all others earning nothing, logarithm of zero income is not defined.

However, since we are working with income power, inequality should mean inequality of income power. The maximum inequality would, then, be attained when **(n - 1)** persons have income power **0**, (or equivalently income **1**), the richest person having the rest of the income, $[1 + n(\mu - 1)]$.

The maximum value attained by **CH** in this case is,

$$CH_{max} = (1/\mu)(n-1)(\mu - 1)\log M \qquad \ldots \ldots \ldots (16)$$

which can be approximated by $(n - 1)\log M$ for large μ.

The proof is very simple.

We have

$$CH = (1/n\mu)\Sigma\, y_i \log y_i - \log M$$

If $y_i = 1$ (or $\log y_i = 0$) for all $i < n$ and

$y_n = [1 + n(\mu - 1)]$ then

$$\log M = (1/n)\log [1 + n(\mu - 1)] \text{ and}$$

$$\begin{aligned} CH &= (1/n\mu)[1 + n(\mu - 1)]\log [1 + n(\mu - 1)] - (1/n\mu)\mu\log [1 + n(\mu - 1)] \\ &= (1/n\mu)[1 + n\mu - n - \mu]\log[1 + n(\mu - 1)] \\ &= (1/n\mu)(n - 1)(\mu - 1)\log [1 + n(\mu - 1)] \\ &= (1/\mu)(n - 1)(\mu - 1)\log M \\ &= CH_{max} \end{aligned}$$

The measure also has a simple relation with the pair of measures proposed by Theil (1967). It equals the sum , **(T + L)**, of the two measures.

We have

$$\begin{aligned} CH &= (1/n\mu)\Sigma\, y_i \log y_i - \log M \\ &= (1/n)\Sigma\, (y_i/\mu)\log(y_i/\mu) + (1/n)\Sigma\, (y_i/\mu)\log \mu - (1/n)\Sigma \log y_i \\ &= [(1/n)\Sigma\, (y_i/\mu)\log(y_i/\mu)] + [(1/n)\Sigma \log(\mu/y_i)] \\ &= T + L \end{aligned} \qquad \ldots \ldots \ldots \ldots (17)$$

where $L = (1/n)\Sigma \log(\mu/y_i)$ is Theil's Second Measure.

This is not the only measure which is the sum of two other distinct measures. The measures Logarithmic variance $\mathbf{H_1^2}$ used by Atkinson (1970) and Stark (1972), Variance of Income Power $\mathbf{H^2}$ and Theil's second measure **L** are related as

$$\mathbf{H^2 = H_1^2 + L}$$

where

$$\mathbf{H_1^2 = (1/n)\ \Sigma\ (\log y_i - \log \mu)^2}$$
$$\mathbf{H^2 = (1/n)\ \Sigma\ (\log y_i - \log M)^2} \text{ and}$$
$$\mathbf{L = (1/n)\ \log (\mu/M)}$$

The measure **CH**, like **T** and **L**, can be computed from the information contained in the Lorenz curve of the income profile. Like **T** and **L**, this measure conforms to the Pigou-Dalton Axiom, Principles of Diminishing Transfers and Population Replication.

Alternative Set of Weights 3

Since we are working with income power, and not income as such, we may think of taking the weights proportional to the gap between the total income power, $\mathbf{\Sigma\log y_i = n\log M}$, in the observed income profile and the income power realised by the individual. This suggests the set of weights

$$\mathbf{w_{4i} = A_4(n\log M - \log y_i)} \qquad \dots\ \dots\ \dots \quad (18)$$

leading to the social welfare function

$$\mathbf{W_4(Y) = A_4\Sigma\ (n\log M - \log y_i)y_i}$$
$$\mathbf{= A_4 n\mu[(n - 1)\log M - CH]}$$

If $y_i = \mu$ for each **i** then **CH = 0** and

$$\mathbf{W_4(Y) = A_4 n(n - 1)\mu\log M = \mu}, \text{ by Axiom N , implying}$$
$$\mathbf{A_4 = 1/[n(n - 1)\log M]} \text{ and}$$
$$\mathbf{W_4(Y) = \mu[1 - CH/\{(n - 1)\log M\}]} \qquad \dots\ \dots\ \dots \quad (19)$$

which is a decreasing function of normalised **CH** measure for large μ .

Sense of relative deprivation contained in the set of weights $\mathbf{w_{4i}}$ suffer from deficiencies similar to those in $\mathbf{w_{1i}}$. Moreover, the desired level of income power

$n \log M$ is not even conceptually attainable by any individual. He can at best have all the income **$n\mu$** leaving nothing for others. His income power would then be **$\log n\mu$** which falls short of **$n \log M$**.

5. AGGREGATE WELFARE AS WEIGHTED SUM OF INCOME POWERS

We have been trying to characterise the sense of relative deprivation of the individual by comparing his income power with some standard income power *aspired for.* Since in this we are working with the transformed variable, **$\log y$**, instead of income as such it appears plausible to consider the distribution of income power, **$\log y$**, instead of income as such. Social welfare should, then, be measured as an weighted sum of income powers. We therefore replace the equation (2) by

$$W(\log Y) = \Sigma\, w_i \log y_i \qquad \text{... ... ... (20)}$$

and read the axiom N as Axiom N^*.

Axiom N^* (Normalisation)
If distribution of income power is perfectly egalitarian so that each $\log y_i = \log M$, then $W(\log Y)$ equals $\log M$.

If every body is enjoying the same income power then that income power alone should be an adequate measure of social welfare.

Social welfare functions corresponding to the set of weights (13) would now become

$$W_5(\log Y) = A_5 \Sigma\, (\log Me - \log y_i)\log y_i$$

$$= A_5 n[\log M(\log Me - \log M) - H^2]$$

If each $\log y_i = \log M$, then $H^2 = 0$ and

$$W_5(\log Y) = A_5 n \log M(\log Me - \log M) = \log M \text{ by Axiom } N^*$$

implying

$$A_5 = 1/[n(\log Me - \log M)] \quad \text{and}$$

$$W_5(\log Y) = \log M(1 - H^2) \qquad \text{... ... ... (21)}$$

The welfare function corresponding to the set of weights (18) would be

$$\mathbf{W_6(logY) = A_6\Sigma\ (nlog\ M - log\ y_i)log\ y_i}$$
$$\mathbf{= A_6n[(n - 1)(log\ M)^2 - H^2]}$$

If each $\mathbf{log\ y_i = log\ M}$, then $\mathbf{H^2 = 0}$ and

$$\mathbf{W_6(logY) = A_6n(n - 1)(log\ M)^2 = log\ M}, \text{ by Axiom } N^*,$$

implying $\mathbf{A_6 = 1/[n(n - 1)log\ M]}$ and

$$\mathbf{W_6(log\ Y) = log\ M[1 - H^2/\{(n - 1)(log\ M)^2\}]} \qquad \text{... ... ... (22)}$$

The function $\mathbf{W_5(logY)}$ is a decreasing function of the Variance of Income Power while $\mathbf{W_6(logY)}$ is that of the normalised Variance of Income Power. The other two sets of weights (9) and (10) do not correspond to any popular measure of inequality.

6. GENESIS OF THE WELFARE MEASURES

Pareto Principle and Population Repluication

Let us confine our attention to the six welfare measures derived by choosing six different set of weights conforming to the axioms **M, E, D** and **N** viz. $\mathbf{W_1, W_2, W_3, W_4, W_5}$ and $\mathbf{W_6}$. The weight attached to income $\mathbf{y_i}$ (or income power $\mathbf{log\ y_i}$) in getting the weighted sum $\Sigma\ \mathbf{w_iy_i}$ (or $\Sigma\ \mathbf{w_ilog\ y_i}$) is proportional to the shortfall of income power realised from the income power *aspired for.* This makes the sense of relative deprivation proportional to the gap between the *desired position* and the *position realised.* The position realised corresponds to the realised income power $\mathbf{log\ y_i}$. Different choice of the desired position leads to different welfare measures. Some of these are decreasing functions of some measures of inequality in popular use.

If the desired position corresponds to having all the income of the society leaving nothing for others the corresponding welfare function turns out to be a function of some measure of inequality normalised by its maximum attainable value such as

$$\mathbf{W_1(Y) = \mu[1 - T/log\ n]}$$
$$\mathbf{W_4(Y) = \mu[1 - CH/\{(n-1)log\ M\}]}$$
$$\mathbf{W_6(log\ Y) = log\ M[1 - H^2/\{(n-1)(log\ M)^2\}]}$$

The welfare measures in this set fail to conform to the PPR. This is because the upper bound of the corresponding measures of inequality depends on the number **n** of the income recipients. When normalised these measures fail to conform to PPR.

When the aspired level of income is lowered suitably the resulting welfare measures become functions of the corresponding measures of inequality as such, not normalised.

$$\mathbf{W_2(Y) = \mu(1 - T)}$$

$$\mathbf{W_3 = \mu(1 - CH)}$$

$$\mathbf{W_5(\log Y) = \log M(1 - H^2)}$$

These measures conform to PPR since the measures of inequality do so. Conforming to PPR is perhaps, a desirable property of a welfare measure. However, as we shall see presently, in lowering the level of aspired income the resulting welfare measure looses another *desirable* property. They fail to conform to what is known as the Paretian Principle (PP).

A view shared by many scholars is that **W(Y)** should be an increasing function of each $\mathbf{y_i}$. If one income is raised without changing the rest of the incomes the aggregate welfare should rise (at least not fall). Such a measure is said to conform to Paretian Principle[7] (PP).

The measures suggested by Sen (1974), Kakwani (1980, 1981), Chakravarty (1982) and those in the first set - $\mathbf{W_1(Y)}$, $\mathbf{W_4(Y)}$ and $\mathbf{W_6(\log Y)}$ which are functions of some normalised measure of inequality, conform to this Paretian Principle.

The welfare measures in the second set - $\mathbf{W_2(Y)}$, $\mathbf{W_3(Y)}$ and $\mathbf{W_5(\log Y)}$ which are functions of the measure of inequality as such, not normalised, fail to conform to PP. The weights in the first set while decreasing, remain non negative throughout and tend

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[7] This principle has recently been criticisedby some scholars. Even *"when the efficiency gain is entirely enjoyed by the richest person, welfare improves"* according to this principle. *"It is difficult to justify this Pareto improvement axiom"* (Tam and Zhang 1996).

to **0** when all incomes tend to accrue in one hand. In the second set weights decrease at a faster rate reaching **0** at a value of **y** corresponding to the aspired level of income power and becomes negative thereafter. Thus, all incomes larger than this value will have a negative weight attached to it. Consequently any further rise in any of these incomes will cause a fall in the aggregate welfare so that these measures fail to conform to PP.

Range of Values

All these welfare measures equal **μ** (or **log M**) when the distribution is perfectly egalitarian. Aggregate welfare is lowered when the distribution becomes unequal. In the extreme case when all incomes (or income power) accrue in one hand, leaving nothing for others, the measures in the first set tend to **0**, while those in the second set assume negative values. The maximum and minimum values attainable by these welfare measures are shown below.

Social welfare functions in the first set assume non-negative values only, while those in the second set can assume negative values as well when inequality is high.

Range of values attainable by various social welfare functions

Social welfare functions	Value attained when the distribution is	
	perfectly egalitarian	Most unequal
$\mu[1-(T/\log n)]$	μ	0
$\mu[1-T]$	μ	$\mu(1-\log n)$
$\mu[1-[CH/\{(n-1)\log M\}]$	μ	0
$\mu[1-CH]$	μ	$\mu[1-(n-1)\log M]$
$\log M[1-\{H^2/(n-1)(\log M)^2$	log M	0
$\log M[1-H^2]$	log M	$\log M[1-(n-1)(\log M)^2]$
$\mu(1-C^2)$	μ	$\mu(2-n)$
$\mu[1-C^2/(n-1)]$	μ	0

Social welfare function obtained by making one's sense of deprivation proportional to the gap between realised and aspired income also has similar properties. If the aspired income is **nμ**, the implied social welfare function is,

$$\mathbf{W_7(Y) = A_7\Sigma\,(n\mu - y_i)y_i}$$
$$\mathbf{= A_7n^2\mu^2 - A_7\Sigma\,(y_i)^2}$$
$$\mathbf{= A_7n^2\mu^2 - A_7n\mu^2[(1/n)\Sigma\,(y_i/\mu)^2 - 1 + 1]}$$
$$\mathbf{= A_7n\mu^2[(n - 1) - C^2]}$$

If each $\mathbf{y_i = \mu}$, then $\mathbf{C^2 = 0}$ and

$\mathbf{W_7(Y) = A_7n\mu^2 = \mu}$, by Axiom N , implying

$\mathbf{A_7 \ = 1/[n(n - 1)\mu]}$ and

$$\mathbf{W_7(Y) = \mu[1 - C^2/(n - 1)]} \qquad \text{... ... ... (23)}$$

Lowering the level of aspired income to **2μ** we get[8],

$$\mathbf{W_8(Y) = A_8\Sigma\,(2\mu - y_i)y_i}$$
$$\mathbf{= \mu(1 - C^2)} \qquad \text{... ... ... (24)}$$

The measure $\mathbf{W_7}$ belongs to the first set while $\mathbf{W_8}$ belongs to the second.

Interpreting the Individual Terms

The individual terms in the weighted sum have often been interpreted as the level of welfare of the individuals, and the aggregate welfare measured as the sum of these individual welfares. Conceptually, individual welfares are necessarily non-negative and should be non-decreasing in income. If income $\mathbf{y_i}$ of individual **i** rises his level of welfare should rise (at least not fall). The welfare measures in the second set fail to conform to this when the individual terms are interpreted as above. However, an alternative interpretation may be given to these terms.

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[8] Blackorby and Donaldson (1978) in a different formulation obtained the social welfare function corresponding to the Coefficient of Variation as

$$\mathbf{W_c(Y) = (1/n)\,\Sigma\,y_i - [(1/n)\,\Sigma\,(y_i - \mu)^2]^{1/2}}$$

which simplifies to

$$\mathbf{W_c(Y) = \mu(1 - C)}$$

The $\mathbf{i}^{th}$ term in the weighted sum may be taken as the *contribution* of income $\mathbf{y}_i$ to the aggregate welfare. This contribution need not necessarily be the same as the level of welfare of the individual **i** in the given income profile. ***If an individual is too rich his contribution to aggregate welfare is negative, though he himself may be at a positive level of welfare.*** How rich is too rich is, of course, a matter of value judgments. The set of weights in the second set of measures suggest that a person is too rich if his income exceeds the level of income aspired for by the typical member of the community.

For a person who is already too rich the level of income *aspired for* will be lower than his realised income. It is not therefore, plausible to expect that he will really aspire for this lower level of income. The problem can be resolved by inviting the *ethical observer* to evaluate the contribution of each person to aggregate welfare. Different ethical observers would then evaluate the contributions imposing his own criterion leading to different social welfare functions.

Choosing a Set of Weights

Like various ways of conceptualisation and measurement of inequality there are numerous ways of characterising sense of relative deprivation and hence choosing the set of weights each leading to a different social welfare function. Since there is no unique *natural* way of choosing the weights, lack of consensus on the exact functional form of the measure of aggregate welfare to be adopted cannot be avoided. Then how does one choose one set of weights out of so many? This is a difficult question yet to be answered. Scholars have proposed certain sets of weights through a set of plausible axioms and then demonstrated relevance of the set of weights chosen to one or the other measure of inequality such as **G, T, CH,** $\mathbf{H}^2$ and $\mathbf{C}^2$. The problem of choice remains.

However, in social sciences criteria such as, intuitive appeal, easy interpretation, have often been used for choosing one among many options. One may, then, choose a social welfare function out of the many alternatives on considerations of the value judgments incorporated in the measure of inequality and the set of weights involved.

Each of these measures have a built in value judgement incorporated through the choice of the set of weights. Judging by these value judgements and consequent properties none seem to have a conspicuous advantage over the others. If one decides to measure social welfare as a function of any of the measure **T, CH, H^2** and **C^2** then the choice between a Social welfare function in the set **(W_1, W_4, W_6, W_7)** and the corresponding Social welfare function in **(W_2, W_3, W_5, W_8)** is really a choice between the *Paretian Principle* and the *Principle of Population Replication.*

Problems in Empirical Estimation

The data base for empirical assessment of social welfare comes generally from sample surveys of income receiving units (households, earners, persons etc.). These data in published form provide percentage of persons and their average income for various income intervals. At times these are available in the form of income shares or average incomes of various deciles or percentiles of income receiving units.

If the measure used to assess welfare is scale invariant and conforms to PPR, we can safely assume that the total number of income receiving units is 100 and total income is 1 keeping the estimated value of the measure unchanged. However, if the measure fails to conform to PPR, knowledge of the size of the population will be needed for its estimstion which is not generally available. On this count the welfare measures in the second set which are decreasing functions of the inequality measures as such will be preferred in empirical analyses.

The measures proposed by Sen (1974) and Kakwani (1980, 1981) are decreasing functions of the inequality measure Gini. Since the upper bound of Gini is independent of the size of the population the normalised measure is also Gini. The welfare measures based on Gini will thus conform to PPR since Gini does so. The upper bounds of T, C^2, H^2 and CH depend on the size of the population while the measures conform to PPR. These when normalised by the respective upper bounds the resulting measures will fail to conform to PPR. Any welfare measure which are decreasing functions of these normalised measures will consequently fail to conform to PPR. It is difficult to have empirical estimates of these measures.

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APPENDIX

Equation (8)

$$W_1(Y) = A_1\Sigma \log(n\mu/y_i)y_i$$

$$= A_1\Sigma (\log n\mu - \log y_i)y_i$$

$$= A_1n\mu\log n\mu - A_1\Sigma y_i\log y_i$$

$$= A_1n\mu\log n\mu - A_1n\mu[(1/n)\Sigma (y_i/\mu)\log (y_i/\mu)$$

$$+(1/n)\Sigma (y_i/\mu)\log \mu]$$

$$= A_1n\mu\log n\mu - A_1n\mu T - A_1n\mu(1/n)\log \mu\Sigma(y_i/\mu)$$

$$= A_1n\mu\log n\mu - A_1n\mu T - A_1n\mu\log \mu$$

$$= A_1n\mu[\log n - T]$$

If each $y_i = \mu$ then $T = 0$ and

$W_1(Y) = A_1n\mu\log n = \mu$, by Axiom N , implying

$A_1 = (1/n\log n)$ and

$$W_1(Y) = \mu[1 - T/\log n] \qquad \text{................(8)}$$

Equation (11)

$$W_2(Y) = A_2\Sigma (\log \mu e - \log y_i)y_i$$

$$= A_2\Sigma (1 + \log \mu - \log y_i)y_i$$

$$= A_2n\mu + A_2n\mu\log \mu - A_2\Sigma y_i\log y_i$$

$$= A_2n\mu + A_2n\mu\log \mu - A_2n\mu[(1/n)\Sigma (y_i/\mu)$$

$$\log (y_i/\mu)] + A_2n\mu(1/n)\log \mu \Sigma(y_i/\mu)$$

$$= A_2n\mu + A_2n\mu\log \mu - A_2n\mu T - A_2n\mu\log \mu$$

$$= A_2n\mu(1-T)$$

If each $y_i = \mu$ then $T = 0$ and

$W_2(Y) = A_2n\mu = \mu$ by Axiom N , implying

$A_2 = (1/n)$ and

$$W_2(Y) = \mu(1 - T) \qquad \text{...............(11)}$$

Equation (14)

$$W_3(Y) = A_3\Sigma\ (\log Me - \log y_i)y_i$$
$$= A_3\Sigma\ (1 + \log M - \log y_i)y_i$$
$$= A_3 n\mu - A_3\Sigma\ (\log y_i - \log M)y_i$$
$$= A_3 n\mu[1 - (1/n\mu)\Sigma\ (\log y_i - \log M)y_i]$$
$$= A_3 n\mu[1 - (1/\mu)\mathrm{Cov}(\log y_i\ ,\ y_i)]$$

If each $y_i = \mu$, then $\mathrm{Cov}(\log y_i\ ,\ y_i) = 0$ and

$W_3(Y) = A_3 n\mu = \mu$, by Axiom N , implying $A_3 = (1/n)$ and

$$W_3(Y) = \mu[1 - (1/\mu)\mathrm{Cov}(\log y_i\ ,\ y_i)]$$
$$= \mu[1 - CH] \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ (14)$$

Equation (19)

$$W_4(Y) = A_4\Sigma\ (n\log M - \log y_i)y_i$$
$$= A_4 n^2\mu\log M - A_4\Sigma\ y_i\log y_i$$
$$= A_4 n^2\mu\log M - A_4 n\mu[(1/n\mu)\Sigma\ y_i\log y_i$$
$$- \log M + \log M]$$
$$= A_4 n^2\mu\log M - A_4 n\mu CH - A_4 n\mu\log M$$
$$= A_4 n\mu[(n - 1)\log M - CH]$$

If $y_i = \mu$ for each i then CH = 0 , and

$W_4(Y) = A_4 n(n - 1)\mu\log M = \mu$, by Axiom N , implying

$A_4 = 1/[n(n - 1)\log M]$ and

$$W_4(Y) = \mu[1 - CH/\{(n - 1)\log M\}] \qquad \ldots\ \ldots\ \ldots\ (19)$$

Equation (21)

$$W_5(\log Y) = A_5\Sigma\ (\log Me - \log y_i)\log y_i$$

$$= A_5 n\log M\log Me - A_5 n[(1/n)\Sigma\ (\log y_i)^2$$

$$- (\log M)^2 + (\log M)^2]$$

$$= A_5 n\log M\log Me - A_5 nH^2 - A_5 n(\log M)^2$$

$$= A_5 n[\log M(\log Me - \log M) - H^2]$$

If each $\log y_i = \log M$, then $H^2 = 0$ and

$$W_5(\log Y) = A_5 n\log M(\log Me - \log M) = \log M \text{ by Axiom } N^*$$

implying

$$A_5 = 1/[n(\log Me - \log M)] \quad \text{and}$$

$$W_5(\log Y) = \log M(1 - H^2) \qquad \ldots\ \ldots\ \ldots\ (21)$$

Equation (22)

$$W_6(\log Y) = A_6\Sigma\ (n\log M - \log y_i)\log y_i$$

$$= A_6 n^2(\log M)^2 - A_6 n[(1/n)\Sigma\ (\log y_i)^2$$

$$- (\log M)^2 + (\log M)^2]$$

$$= A_6 n^2(\log M)^2 - A_6 nH^2 - A_6 n(\log M)^2$$

$$= A_6 n[(n - 1)(\log M)^2 - H^2]$$

If each $\log y_i = \log M$, then $H^2 = 0$ and

$$W_6(\log Y) = A_6 n(n - 1)(\log M)^2 = \log M, \text{ by Axiom } N^*,$$

implying

$$A_6 = 1/[n(n - 1)\log M] \quad \text{and}$$

$$W_6(\log Y) = \log M[1 - H^2/\{(n - 1)(\log M)^2\}] \qquad \ldots.\ (22)$$

Equation (23)

$$W_7(Y) = A_7\Sigma\ (n\mu - y_i)y_i$$
$$= A_7n^2\mu^2 - A_7\Sigma\ (y_i)^2$$
$$= A_7n^2\mu^2 - A_7n\mu^2[(1/n)\Sigma\ (y_i/\mu)^2 - 1 + 1]$$
$$= A_7n\mu^2[(n - 1) - C^2]$$

If each $y_i = \mu$, then $C^2 = 0$ and

$W_7(Y) = A_7n\mu^2 = \mu$, by Axiom N , implying

$A_7 = 1/[n(n - 1)\mu]$ and

$$W_7(Y) = \mu[1 - C^2/(n - 1)] \qquad \ldots\ \ldots\ \ldots\ \ldots\ (23)$$

Equation (24)

$$W_8(Y) = A_8\Sigma\ (2\mu - y_i)y_i$$
$$= 2A_8n\mu^2 - A_8\Sigma\ (y_i)^2$$
$$= 2A_8n\mu^2 - A_8n\mu^2[(1/n)\Sigma\ (y_i/\mu)^2 - 1 + 1]$$
$$= 2A_8n\mu^2 - A_8n\mu^2C^2 - A_8n\mu^2$$
$$= A_8n\mu^2(1 - C^2)$$

If each $y_i = \mu$, then $C^2 = 0$ and

$W_8(Y) = A_8n\mu^2 = \mu$, by Axiom N , implying

$A_8 = 1/(n\mu)$ and

$$W_8 = \mu(1 - C^2) \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ (24)$$